4.2 Energy and Power

Definition 4.12. For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized (instantaneous) power**.

Definition 4.13. The total (normalized) ***energy*** of a signal $g(t)$ is given by

$$E_g = \int_{-\infty}^{+\infty} p_g(t)dt = \int_{-\infty}^{+\infty} |g(t)|^2 \, dt = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt.$$

4.14. By the **Parseval's theorem** discussed in 2.43, we have

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

Definition 4.15. The average (normalized) ***power*** of a signal $g(t)$ is given by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g\,(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt.$$

Definition 4.16. To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

(a) The "**time-average**" operator:

$$\langle g \rangle \equiv \langle g\,(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{g}\,(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{g}\,(t)dt \qquad (42)$$

(b) The **inner-product** operator:

$$\langle g_1, g_2 \rangle \equiv \langle g_1\,(t)\,, g_2\,(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \qquad (43)$$

4.17. Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$ where $G = \mathcal{F}\{g\}$
- $P_g = \left\langle |g|^2 \right\rangle$

- Parseval's theorem: $\langle g_1, g_2\rangle = \langle G_1, G_2\rangle$
  where $G_1 = \mathcal{F}\{g_1\}$ and $G_2 = \mathcal{F}\{g_2\}$

4.18. Time-Averaging over Periodic Signal: For ***periodic*** signal $g(t)$ with period T_0, the time-average operation in (42) can be simplified to

$$\langle g\rangle = \frac{1}{T_0}\int_{T_0} g(t)dt$$

where the integration is performed over a period of g.

Example 4.19. $\langle \cos(2\pi f_0 t + \theta)\rangle =$

Similarly, $\langle \sin(2\pi f_0 t + \theta)\rangle =$

Example 4.20. $\left\langle \cos^2(2\pi f_0 t + \theta)\right\rangle =$

Example 4.21. $\left\langle e^{j(2\pi f_0 t+\theta)}\right\rangle = \langle \cos(2\pi f_0 t + \theta) + j\sin(2\pi f_0 t + \theta)\rangle$

Example 4.22. Suppose $g(t) = ce^{j2\pi f_0 t}$ for some (possibly complex-valued) constant c and (real-valued) frequency f_0. Find P_g.

4.23. When the signal $g(t)$ can be expressed in the form $g(t) = \sum_k c_k e^{j2\pi f_k t}$ and the f_k are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

Example 4.24. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j8\pi t}$. Find P_g.

Example 4.25. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$. Find P_g.

Example 4.26. Suppose $g(t) = \cos(2\pi f_0 t + \theta)$. Find P_g.

Here, there are several ways to calculate P_g. We can simply use Example 4.20. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

4.27. The (average) power of a sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2 \cos^2\theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g}\cos(2\pi f_0 t + \theta).$$

4.28. Extension of 4.27: Consider sinusoids $A_k \cos(2\pi f_k t + \theta_k)$ whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2}\sum_k |A_k|^2.$$

Example 4.29. Suppose $g(t) = 2\cos\left(2\pi\sqrt{3}t\right) + 4\cos\left(2\pi\sqrt{5}t\right)$. Find P_g.

Example 4.30. Suppose $g(t) = 3\cos(2t) + 4\cos(2t - 30°) + 5\sin(3t)$. Find P_g.

4.31. For ***periodic*** signal $g(t)$ with period T_0, there is also no need to carry out the limiting operation to find its (average) power P_g. We only need to find an average carried out over a single period:

$$P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt.$$

Example 4.32.

4.33. When the Fourier series expansion (to be reviewed in Section 4.3) of the signal is available, it is easy to calculate its power:

(a) When the corresponding Fourier series expansion $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$ is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

(b) When the signal $g(t)$ is real-valued and its (compact) trigonometric Fourier series expansion $g(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle\phi_k)$ is known,

$$P_g = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2.$$

Definition 4.34. Based on Definitions 4.13 and 4.15, we can define three distinct classes of signals:

(a) If E_g is finite and nonzero, g is referred to as an ***energy signal***.

(b) If P_g is finite and nonzero, g is referred to as a ***power signal***.

(c) Some signals[17] are neither energy nor power signals.

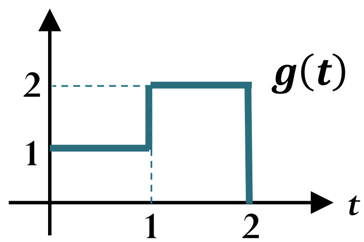
- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are disjoint.

Example 4.35. Rectangular pulse

[17] Consider $g(t) = t^{-1/4} 1_{[t_0,\infty)}(t)$, with $t_0 > 0$.

Example 4.36. Sinc pulse

Example 4.37 (M2018)**.** Consider a signal $g(t)$ below. Note that $g(t) = 0$ outside of the interval $[0, 2]$.



Let

$$y(t) = \sum_{k=-\infty}^{\infty} g(t-k) \quad \text{and} \quad z(t) = \sum_{k=-\infty}^{\infty} g(t-2k).$$

Calculate the following quantities:

(a) energy E_g

(b) average power P_g

(c) $\langle g(t) \rangle$

(d) average power P_y

(e) average power P_z

The table below summarizes, for each signal, its (i) time average (ii) (total) energy, (iii) (average) power, and indication (by putting a Y (for a yes) or an N (for a no)) in part (iv) whether it is an energy signal and in part (v) whether it is a power signal.

		$g(t)$	$y(t)$	$z(t)$
(i)	$\langle\cdot\rangle$			
(ii)	(Total) Energy			
(iii)	(Average) Power			
(iv)	Energy Signal?			
(v)	Power Signal?			

Example 4.38. For $\alpha > 0$, $g(t) = Ae^{-\alpha t}1_{[0,\infty)}(t)$ is an energy signal with $E_g = |A|^2/2\alpha$.

Example 4.39. The rotating phasor signal $g(t) = ce^{j(2\pi f_0 t+\theta)}$ is a power signal with $P_g = |c|^2$.

Example 4.40. The sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is a power signal with $P_g = |A|^2/2$.

4.41. Consider the transmitted signal

$$x(t) = m(t)\cos(2\pi f_c t + \theta)$$

in DSB-SC modulation. Suppose $M(f - f_c)$ and $M(f + f_c)$ do not overlap (in the frequency domain).

(a) If $m(t)$ is a power signal with power P_m, then the average transmitted power is

$$P_x = \frac{1}{2}P_m.$$

- Q: Why is the power (or energy) reduced?

- Remark: When $x(t) = \sqrt{2}m(t)\cos(2\pi f_c t + \theta)$ (with no overlapping between $M(f - f_c)$ and $M(f + f_c)$), we have $P_x = P_m$.

(b) If $m(t)$ is an energy signal with energy E_m, then the transmitted energy is

$$E_x = \frac{1}{2}E_m.$$

Example 4.42. Suppose $m(t) = \cos(2\pi f_c t)$. Find the average power in $x(t) = m(t)\cos(2\pi f_c t)$.